Scaling properties of Lyapunov spectra for the band random matrix model

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The transfer-matrix method is applied to quasi-one-dimensional disordered media described by a one-dimensional tight-binding Hamiltonian with long-range random interactions. We investigate the scaling properties of the whole Lyapunov spectrum in the limit of the interaction range b tending to infinity. Two different energy dependencies are found around the maximum and the minimum Lyapunov exponents. Moreover, a singular behavior in the lower part of the Lyapunov spectrum is found at the band edge. Finally, scaling properties of the fluctuations are also analyzed. [S1063-651X(96)51406-6]

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Lyapunov exponents represent a powerful tool for the investigation of chaotic properties in nonlinear dynamical systems and of localization properties in disordered systems. In the former case, one is interested in the evolution in tangent space, while in the latter, the spatial structure of the eigenfunctions is studied by means of a transfer-matrix approach.

While many results are available in low-dimensional spaces [i.e., strange attractors and strictly one-dimensional (1D) disordered systems], much less clear is the situation in spatially extended dynamical systems or 2D-3D disordered systems. In that case, the concept of Lyapunov spectrum is introduced and its scaling properties are investigated [1,2]. Yet another class of disordered systems which require considering high-rank matrices is that of quasi-1D or 1D systems with long-range hopping. The simplest and general model in this class is represented by the Schrödinger equation with interactions described by band random matrices (BRMs). This ensemble is defined as the set of real symmetric matrices the entries of which are independent Gaussian variables with zero average and variance $\sigma^2 = 1+\delta_{n,m}$ ($\delta_{n,m}$ is the Kronecker symbol) if $|n-m| \leq b$ and zero otherwise. The parameter b defines the hopping range between neighboring sites and, in the quasi-1D interpretation, is the number of transverse channels along a thin wire.

Extensive numerical [3] and analytical [4] studies allowed clarifying the statistical properties of the eigenstates. In all the above studies, the interest was concentrated only on the minimun Lyapunov exponent although it is by now well understood that quantum transport properties in disordered systems involve also the other Lyapunov exponents [2]. In this paper, by following a transfer-matrix approach, we study the structure of the whole Lyapunov spectrum showing that it approaches an asymptotic shape as $b \to \infty$. Two different energy dependencies are found for the upper and for the lower part of the spectrum, respectively. Furthermore, we numerically show the existence of an anomalous scaling at the band edge of the spectrum of the Schrödinger operator. Finally, we study the scaling behavior of the fluctuations of the Lyapunov exponents.

Our starting point is the time-dependent Schrödinger equation

$$i\frac{dc_n(t)}{dt} = \sum_{m=n-b}^{n+b} H_{n,m} c_m , \tag{1}$$

written in dimensionless units. The variable $c_n(t)$ is the probability amplitude for an electron to be at site n, while $H_{n,m}$ is a symmetric BRM. The eigenvalues can be obtained by substituting the assumption $c_n(t) = \exp(-iEt)\psi_n$ in Eq. (1) and solving the resulting equation for ψ_{n+b},

$$\psi_{n+b} = \frac{1}{H_{n,n+b}} \left(E\psi_n - \sum_{m=n-b}^{n+b-1} H_{n,m}\psi_m \right). \tag{2}$$

By defining $x_n(i) \equiv \psi_{n+b-i}$, the above equation can be recasted in the form of a $2b$-dimensional linear map T_n,

$$x_{n+1}(1) = \frac{1}{H_{n,n+b}} \left(E x_n(b) - \sum_{j=1}^{2b} H_{n,n+b-j} x_n(j) \right),$$

$$x_{n+1}(j) = x_n(j-1) \quad 1 < j \leq 2b. \tag{3}$$

Recursive relation (3) is characterized by $2b$ Lyapunov exponents. Let us denote with $\lambda_1(\nu,n) > \lambda_2(\nu,n) > \cdots > \lambda_i(\nu,n) > \cdots > \lambda_{2b}(\nu,n)$ the effective (finite-time) Lyapunov exponents [5,6] computed over a number n of iterations at energy $E = \nu\sqrt{b}$. Here, ν denotes the rescaled energy, the density of which is known to follow the semicircle law, i.e., $\rho(\nu) = (8-\nu^2)b^2/4\pi$ [7].

In analogy with the 2D Anderson problem on a stripe of width b, we deal with $2b \times 2b$ matrices; however, here, the single matrix T_n is not symplectic and the determinant is not even equal to 1 but to $(-1)^{2b+1}(-H_{b+1,1}/H_{b+1,2b+1})$. However, statistical invariance of the disorder under space-reversal guarantees that the product of infinitely many matrices preserves volumes and ensures the symmetric structure of Lyapunov spectra typical of the symplectic ensemble. More precisely, the Lyapunov exponents are arranged in b

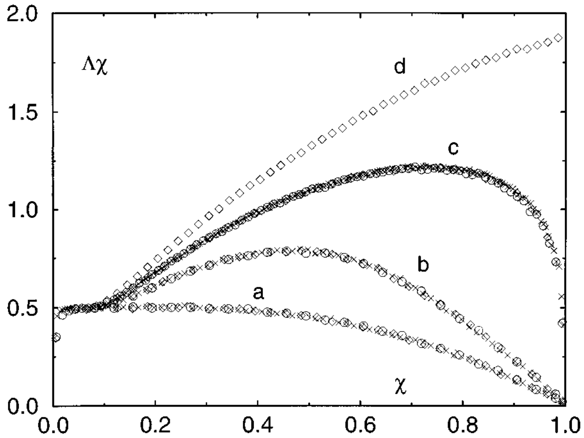

FIG. 1. Lyapunov spectra for energy densities $\nu=0$, 1.8, ν_e, 4 (curves a, b, c, and d, respectively). Circles, diamonds, and crosses correspond to $b=80$, 100, and 200, respectively. The superposition of the curves for different b values strongly suggests the existence of a limit distribution. The curve for $\nu=4$ is reported only for the sake of comparison, the energy being out of the spectrum. All curves start from $\Lambda\chi\simeq 0.346$, value corresponding to $\lambda_1\simeq 0.693$ [from Eq. (4), $\lambda_1=2\Lambda(1/2b)$].

pairs $[\lambda_i(\nu)=-\lambda_{2b-i+1}(\nu)]$. For this reason, in what follows we always report only the positive exponents.

A further analogy exists between Eq. (2) and the evolution equation in tangent space for dynamical systems with delayed feedback [8], where the delay time plays a role equivalent to b in controlling the ''range'' of interactions. These analogies suggest the following scaling relations for the Lyapunov spectrum:

$$\lambda_i(\nu)=\Lambda(\chi,\nu)/b, \quad \chi=(i-1/2)/b, \tag{4}$$

where $i=1,\ldots,b$ and the correction term 1/2 is added to guarantee that the spectrum is symmetric around $\chi=1$, independently of b [9].

The numerical results for three different energy densities, $\nu=0$, 1.8, and $\nu_e=\sqrt{8}$ (band edge), are reported in Fig. 1 (curves a, b, and c, respectively) by referring to the variable $\Lambda\chi$ to get rid of the singularity around $\chi=0$. The points corresponding to the same energy but different values of b fall onto the same smooth curve with a good accuracy [10], confirming the scaling hypothesis. The divergence of $\Lambda(0,\nu)$ follows from the existence of an exponent which remains finite for $b\to\infty$. This is related to the presence of ''short-time'' interactions [8], i.e., the explicit dependence of ψ_{n+1} on ψ_n, as it can be seen from Eq. (2).

An additional interesting feature of the Lyapunov spectrum that we found is that $\Lambda(\chi,\nu)$ is independent of ν at small χ values (i.e., large Λ) even for energies outside the semicircle as clearly demostrated in Fig. 1, where curve (d) corresponds to $\nu=4$. Our data indicate that this universal scaling behavior of the upper part of the spectrum holds up to $\chi\simeq 0.1$. In particular, the value of the maximum Lyapunov exponent is equal to $0.693\ldots$. The independency of the energy can be understood with a self-consistency argument. From Eq. (2), if we assume that $\psi_n\simeq e^{n\lambda_1}$ with λ_1 independent of b, it is obvious that the energy dependence is asymptotically $(b\to\infty)$ negligible as it arises from a term $e^{-b\lambda_1}$ times smaller than the leading term [the one on the left hand side of Eq. (2)]. Thus, in the limit $b\to\infty$, the maximum Lyapunov exponent is expected to be the same as that for the recursive relation

$$\psi_n\simeq \frac{1}{H_{j,n}}\sum_{m=j+1}^{n-1} H_{j,m}\psi_m, \tag{5}$$

where all matrix elements are independent, identically distributed variables. An analytical estimate for λ_1 can be obtained from the following simple argument. Each coefficient $H_{j,m}/H_{j,n}$ is equally likely larger and smaller than 1 in absolute value, so that we can assume that it is equal to 1 on the average. By substituting this assumption in Eq. (5) and summing the corresponding series, one finds that for $n\to\infty$, $\lambda_1=\ln 2$. This result is astonishingly close to our numerical findings although we have neglected both fluctuations of the coefficients and interference effects due to the random sign of each ratio $H_{j,m}/H_{j,n}$.

The localization length $l_\infty(\nu)$ of the eigenfunction of energy ν is known to be the inverse of the minimum positive Lyapunov exponent $\lambda_{min}(\nu)$ [11]. The latter quantity is obtained by setting $\chi=1-1/(2b)$, i.e., $i=b$, in the spectrum $\Lambda(\chi)$. Since $\Lambda(1)=0$ for all energy values in the spectrum (this relation expresses the analytical result that the localization length diverges as b^2 for $b\to\infty$) the localization length can be obtained by linearizing Λ around $\chi=1$ [see also relation (4)],

$$l_\infty(\nu)=-2b^2/\Lambda'(1,\nu). \tag{6}$$

A necessary condition for Eq. (6) to be correct is that the leading finite band-size (b) correction arises from the discreteness of the χ values. This is not *a priori* obvious, since the rate of convergence of the Lyapunov spectrum to its asymptotic shape is a problem that has never been, to our knowledge, quantitatively investigated. In our case, the convergence turns out to be very fast since the numerical analysis reveals a perfect agreement with the theoretical results obtained in [7], where it was rigorously proved that the localization length is $l_\infty(\nu)=b^2(8-\nu^2)/6$.

Moreover, our results suggest, at the band edge, a singular behavior of the spectrum around $\chi=1$,

$$\Lambda(\chi,\nu_e)\simeq(1-\chi)^\beta. \tag{7}$$

A slow dependence on b prevents a direct accurate estimate of the critical exponent β. Such a difficulty has been circumvented with an appropriate conjecture about the main correction term to the leading behavior expressed by Eq. (7). Our ansatz consists in assuming that $\Lambda(\chi,\nu_e)$ is an analytic function of $(1-\chi)^\beta$ for some β value, i.e., that the finite size corrections are proportional to $(1-\chi)^{2\beta}$. Accordingly, β can be determined from a quadratic fit of the spectral tail. More precisely, since we know that $\Lambda(1)=0$, the value of β is obtained by imposing that the fitted spectrum crosses 0 at $\chi=1$. The quality of the fit strongly supports our ansatz and yields $\beta=0.34\pm 0.01$. The numerical results are reported in Fig. 2.

A scaling behavior with a similar exponent $\beta=1/3$ has been already observed in two other physical problems: (i) the

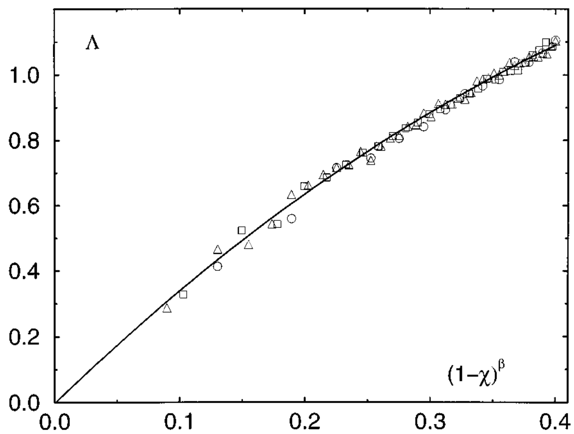

FIG. 2. The Lyapunov spectrum in the vicinity of $\chi=1$ for the energy at the band edge. Circles, squares, and triangles correspond to $b=200$, 400, and 600, respectively. The solid line is the result of a quadratic fitting as a function of $(1-\chi)^\beta$ with $\beta=0.34$.

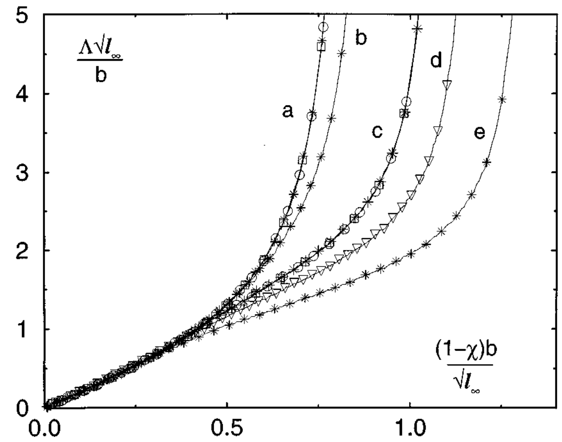

FIG. 3. Rescaled Lyapunov spectra for energies $\nu=0$, 1, 1.8, 2, and 2.2 (curves a, b, c, d, and e respectively). Circles, squares, stars, and triangles correspond to $b=80$, 100, 200, and 300, respectively. The good overlap indicates the existence of a scaling behavior in the region of small Lyapunov exponents for a wide interval of energies.

spatial Lyapunov spectrum for a chain of coupled logistic maps [9]; (ii) the standard temporal Lyapunov spectrum in the 3D evolution of molecules interacting via a Lennard-Jones-type potential [12]. The three systems, *a priori*, share only the symmetry of the spectrum, a property which follows from the reversibility along the (pseudo)time axis. Therefore, it is very likely that a universal mechanism underlies this phenomenon. The anomalous behavior of the Lyapunov exponents arises at the band edge of the spectrum, so that it is related to the structure of the ground state, i.e., to zero-temperature statistical properties.

In the study of localization properties in BRMs, remarkable scaling relations have been found [3,4], which show that many properties can be traced back to the density of states. Therefore, it appears natural to ask whether the shape of the Lyapunov spectrum can be scaled in a uniform way for different energy values. However, our study has revealed the existence of a sort of ‘‘phase transition’’ separating two regimes characterized by different scaling properties. The lower part of the spectrum is well described by the following ansatz (see Fig. 3);

$$\Lambda(\chi,\nu)=\frac{b}{\sqrt{l_\infty(\nu)}}f\left(\frac{b(1-\chi)}{\sqrt{l_\infty(\nu)}}\right) \tag{8}$$

which is in the spirit of previous studies [3,4]. This is the most important part for what concerns applications to solid state physics: localization and conductance in the corresponding disordered system are mainly determined from the properties of the open channels, i.e., small Lyapunov exponents. Notice, for instance, that by assuming a linear dependence of $f(x)$ around $x=0$, as suggested from Fig. 3, Eq. (8) reduces to relation (6) for the localization length. However, the scaling region becomes increasingly small when the band edge is approached. This is consistent with the singular behavior observed therein.

The second scaling regime occurs for $\chi<0.1$ (see Fig. 1), where no energy dependence is observed at all. The above two findings suggest altogether that, at variance with what was found for the 2D Anderson problem [2], a single parameter scaling does not hold for the whole spectrum.

The last problem addressed in this paper concerns sample-to-sample fluctuations of the effective Lyapunov exponents $\lambda_i(\nu,n)$. In the context of low-dimensional strange attractors, this approach leads to the definition of the generalized Lyapunov exponents [5]. It is interesting to investigate the scaling behavior of such fluctuations in high-dimensional systems in connection either with space-time chaos or conductance fluctuations in solid state physics. Here, we limit ourselves to compute the variance of the distribution, which, in low-dimensional cases, is known to be inversely proportional to n. Thus, by recalling the scaling properties of the Lyapunov spectrum [Eq. (4)], it is natural to conjecture that the variance $V_i(\nu,n)$ of the ith effective Lyapunov exponent scales as

$$V_i(\nu,n)=\frac{s^2(\chi,\nu)}{nb^2}. \tag{9}$$

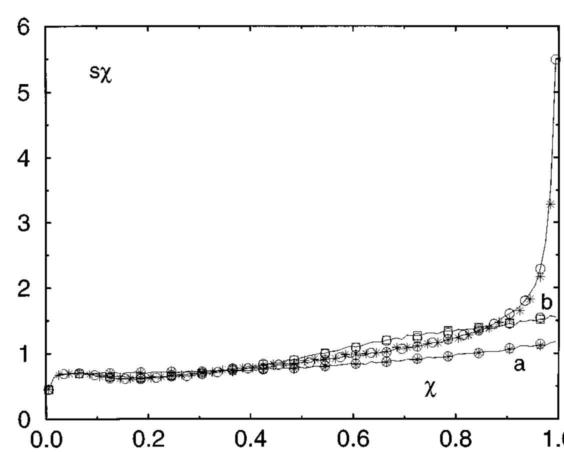

FIG. 4. Scaling of $s(\chi,\nu)$ for energies $\nu=0$, 1.8, and ν_e (curves a, b, and c, respectively). Stars, circles, and squares correspond to $n=150$, 200, and 300, respectively. Notice the divergence occurring at band edge.

The good overlap of the curves reported in Fig. 4 confirms the scaling dependence conjectured in Eq. (9). At variance with Λ, which goes to 0 for $\chi \rightarrow 1$, the scaled standard deviation $s(\chi,\nu)$ remains finite for $\chi \rightarrow 1$. This implies that fluctuations play a prominent role in determining the localization properties of the eigenstates of the Schrödinger operator (1) and extends the previous findings for just the minimum Lyapunov exponent [13]. The larger fluctuations observed at the band edge where $s\chi$ seems to diverge (see curve c in Fig. 4) somehow compensate the slower convergence to zero of the Lyapunov spectrum.

In conclusion, we have investigated the Lyapunov spectra of the Hamiltonian map (2) describing quasi-1D and 1D disordered systems with long-range interactions. We discovered remarkable scaling properties, which appear to be universal [9,12]. Furthermore, we have found two different scalings for the upper and the lower part of the spectrum. This may indicate that local characteristics of the core of eigenstates have different scaling properties in comparison to global characteristics of the shape of the eigenstates. At the end we have investigated the fluctuations of the whole spectrum finding an interesting scaling behavior.

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